

Improved upper bounds for vertex and edge fault diameters of Cartesian graph bundles

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Abstract

Mixed fault diameter of a graph G , $\mathcal{D}_{(a,b)}(G)$, is the maximal diameter of G after deletion of any a vertices and any b edges. Special cases are the (vertex) fault diameter $\mathcal{D}_a^V = \mathcal{D}_{(a,0)}$ and the edge fault diameter $\mathcal{D}_a^E = \mathcal{D}_{(0,a)}$. Let G be a Cartesian graph bundle with fibre F over the base graph B . We show that

(1) $\mathcal{D}_{a+b+1}^V(G) \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$ when the graphs F and B are k_F -connected and k_B -connected, $0 < a < k_F$, $0 < b < k_B$, and provided that $\mathcal{D}_{(a-1,1)}(F) \leq \mathcal{D}_a^V(F)$ and $\mathcal{D}_{(b-1,1)}(B) \leq \mathcal{D}_b^V(B)$ and

(2) $\mathcal{D}_{a+b+1}^E(G) \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B)$ when the graphs F and B are k_F -edge connected and k_B -edge connected, $0 \leq a < k_F$, $0 \leq b < k_B$, and provided that $\mathcal{D}_a^E(F) \geq 2$ and $\mathcal{D}_b^E(B) \geq 2$.

Key words: vertex fault diameter, edge fault diameter, mixed fault diameter, Cartesian graph bundle, Cartesian graph product, interconnection network, fault tolerance.

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1 Introduction

The concept of fault diameter of Cartesian product graphs was first described in [20], but the upper bound was wrong, as shown by Xu, Xu and Hou who provided a small counter example and corrected the mistake [27]. More precisely, denote by $\mathcal{D}_a^V(G)$ the fault diameter of a graph G , a maximum diameter of G after deletion of any a vertices, and $G \square H$ the Cartesian product of graphs G and H . Xu, Xu and Hou proved [27]

$$\mathcal{D}_{a+b+1}^V(G \square H) \leq \mathcal{D}_a^V(G) + \mathcal{D}_b^V(H) + 1$$

while the claimed bound in [20] was $\mathcal{D}_a^V(G) + \mathcal{D}_b^V(H)$. (Our notation here slightly differs from notation used in [20] and [27].) The result was later generalized to graph bundles in [2] and generalized graph products (as defined by [9]) in [28]. Here we show that in most cases of Cartesian graph bundles the bound can indeed be improved to the one claimed in [20].

Methods used involve the theory of mixed connectivity and recent results on mixed fault diameters [6,14,15,16]. For completeness, we also give the analogous improved upper bound for edge fault diameter.

The rest of the paper is organized as follows. In the next section we recall that the graph products and graph bundles often appear as practical interconnection network topologies because of some attractive properties they have. In Section 3 we provide general definitions, in particular of the connectivities. Section 4 introduces graph bundles and recalls relevant previous results. The improved bounds are proved in Section 5.

2 Motivation - interconnection networks

Graph products and bundles belong to a class of frequently studied interconnection network topologies. For example meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some other well-known interconnection network topologies are Cartesian graph bundles, for example twisted hypercubes [10,13] and multiplicative circulant graphs [25].

In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and that the delays in communication must not be too long. Furthermore, an interconnection network should be fault tolerant, because practical communication networks are exposed to failures of

network components. Both failures of nodes and failures of connections between them happen and it is desirable that a network is robust in the sense that a limited number of failures does not break down the whole system. A lot of work has been done on various aspects of network fault tolerance, see for example the survey [9] and the more recent papers [18,26,29]. In particular the fault diameter with faulty vertices, which was first studied in [20], and the edge fault diameter have been determined for many important networks recently [2,3,4,5,11,12,21,27]. Usually either only edge faults or only vertex faults are considered, while the case when both edges and vertices may be faulty is studied rarely. For example, [18,26] consider Hamiltonian properties assuming a combination of vertex and edge faults. In recent work on fault diameter of Cartesian graph products and bundles [2,3,4,5], analogous results were found for both fault diameter and edge fault diameter. However, the proofs for vertex and edge faults are independent, and our effort to see how results in one case may imply the others was not successful. A natural question is whether it is possible to design a uniform theory that covers simultaneous faults of vertices and edges. Some basic results on edge, vertex and mixed fault diameters for general graphs appear in [6]. In order to study the fault diameters of graph products and bundles under mixed faults, it is important to understand generalized connectivities. Mixed connectivity which generalizes both vertex and edge connectivity, and some basic observations for any connected graph are given in [14]. We are not aware of any earlier work on mixed connectivity. A closely related notion is the connectivity pairs of a graph [8], but after Mader [22] showed the claimed proof of generalized Menger's theorem is not valid, work on connectivity pairs seems to be very rare.

Upper bounds for the mixed fault diameter of Cartesian graph bundles are given in [15,16] that in some case also improve previously known results on vertex and edge fault diameters on these classes of Cartesian graph bundles [2,5]. However results in [15] address only the number of faults given by the connectivity of the fibre (plus one vertex), while the connectivity of the graph bundle can be much higher when the connectivity of the base graph is substantial, and results in [16] address only the number of faults given by the connectivity of the base graph (plus one vertex), while the connectivity of the graph bundle can be much higher when the connectivity of the fibre is substantial. An upper bound for the mixed fault diameter that would take into account both types of faults remains to be an interesting open research problem.

3 Preliminaries

A *simple graph* $G = (V, E)$ is determined by a *vertex set* $V = V(G)$ and a set $E = E(G)$ of (unordered) pairs of vertices, called *edges*. As usual, we will use

the short notation uv for edge $\{u, v\}$. For an edge $e = uv$ we call u and v its *endpoints*. It is sometimes convenient to consider the union of *elements* of a graph, $S(G) = V(G) \cup E(G)$. Given $X \subseteq S(G)$ then $S(G) \setminus X$ is a subset of elements of G . However, note that in general $S(G) \setminus X$ may not induce a graph. As we need notation for subgraphs with some missing (faulty) elements, we formally define $G \setminus X$, the subgraph of G after deletion of X , as follows:

Definition 3.1 *Let $X \subseteq S(G)$, and $X = X_E \cup X_V$, where $X_E \subseteq E(G)$ and $X_V \subseteq V(G)$. Then $G \setminus X$ is the subgraph of $(V(G), E(G) \setminus X_E)$ induced on vertex set $V(G) \setminus X_V$.*

A *walk* between vertices x and y is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$ where $x = v_0$, $y = v_k$, and $e_i = v_{i-1}v_i$ for each i . A walk with all vertices distinct is called a *path*, and the vertices v_0 and v_k are called the *endpoints* of the path. The *length* of a path P , denoted by $\ell(P)$, is the number of edges in P . The *distance* between vertices x and y , denoted by $d_G(x, y)$, is the length of a shortest path between x and y in G . If there is no path between x and y we write $d_G(x, y) = \infty$. The *diameter* of a connected graph G , $\mathcal{D}(G)$, is the maximum distance between any two vertices in G . A path P in G , defined by a sequence $x = v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k = y$ can alternatively be seen as a subgraph of G with $V(P) = \{v_0, v_1, v_2, \dots, v_k\}$ and $E(P) = \{e_1, e_2, \dots, e_k\}$. Note that the reverse sequence gives rise to the same subgraph. Hence we use P for a path either from x to y or from y to x . A graph is *connected* if there is a path between each pair of vertices, and is *disconnected* otherwise. In particular, K_1 is by definition disconnected. The *connectivity* (or *vertex connectivity*) $\kappa(G)$ of a connected graph G , other than a complete graph, is the smallest number of vertices whose removal disconnects G . For complete graphs is $\kappa(K_n) = n - 1$. We say that G is *k-connected* (or *k-vertex connected*) for any $0 < k \leq \kappa(G)$. The *edge connectivity* $\lambda(G)$ of a connected graph G , is the smallest number of edges whose removal disconnects G . A graph G is said to be *k-edge connected* for any $0 < k \leq \lambda(G)$. It is well known that (see, for example, [1], page 224) $\kappa(G) \leq \lambda(G) \leq \delta_G$, where δ_G is smallest vertex degree of G . Thus if a graph G is *k-connected*, then it is also *k-edge connected*. The reverse does not hold in general.

The mixed connectivity generalizes both vertex and edge connectivity [14,15]. Note that the definition used in [15] and here slightly differs from the definition used in a previous work [14].

Definition 3.2 *Let G be any connected graph. A graph G is (p, q) -connected, if G remains connected after removal of any p vertices and any q edges.*

We wish to remark that the mixed connectivity studied here is closely related to connectivity pairs as defined in [8]. Briefly speaking, a connectivity pair of a graph is an ordered pair (k, ℓ) of two integers such that there is

some set of k vertices and ℓ edges whose removal disconnects the graph and there is no set of $k - 1$ vertices and ℓ edges or of k vertices and $\ell - 1$ edges with this property. Clearly (k, ℓ) is a connectivity pair of G exactly when: (1) G is $(k - 1, \ell) + \text{connected}$, (2) G is $(k, \ell - 1) + \text{connected}$, and (3) G is not $(k, \ell) + \text{connected}$. In fact, as shown in [14], (2) implies (1), so (k, ℓ) is a connectivity pair exactly when (2) and (3) hold.

From the definition we easily observe that any connected graph G is $(0, 0) + \text{connected}$, $(p, 0) + \text{connected}$ for any $p < \kappa(G)$ and $(0, q) + \text{connected}$ for any $q < \lambda(G)$. In our notation $(i, 0) + \text{connected}$ is the same as $(i + 1) - \text{connected}$, i.e. the graph remains connected after removal of any i vertices. Similarly, $(0, j) + \text{connected}$ means $(j + 1) - \text{edge connected}$, i.e. the graph remains connected after removal of any j edges.

Clearly, if G is a $(p, q) + \text{connected}$ graph, then G is $(p', q') + \text{connected}$ for any $p' \leq p$ and any $q' \leq q$. Furthermore, for any connected graph G with $k < \kappa(G)$ faulty vertices, at least k edges are not working. Roughly speaking, graph G remains connected if any faulty vertex in G is replaced with a faulty edge. It is known [14] that if a graph G is $(p, q) + \text{connected}$ and $p > 0$, then G is $(p - 1, q + 1) + \text{connected}$. Hence for $p > 0$ we have a chain of implications: $(p, q) + \text{connected} \implies (p - 1, q + 1) + \text{connected} \implies \dots \implies (1, p + q - 1) + \text{connected} \implies (0, p + q) + \text{connected}$, which generalizes the well-known proposition that any k -connected graph is also k -edge connected. Therefore, a graph G is $(p, q) + \text{connected}$ if and only if $p < \kappa(G)$ and $p + q < \lambda(G)$.

Note that by our definition the complete graph K_n , $n \geq 2$, is $(n - 2, 0) + \text{connected}$, and hence $(i, j) + \text{connected}$ for any $i + j \leq n - 2$. Graph K_2 is $(0, 0) + \text{connected}$, and mixed connectivity of K_1 is not defined.

If for a graph G $\kappa(G) = \lambda(G) = k$, then G is $(i, j) + \text{connected}$ exactly when $i + j < k$. However, if $2 \leq \kappa(G) < \lambda(G)$, the question whether G is $(i, j) + \text{connected}$ for $1 \leq i < \kappa(G) \leq i + j < \lambda(G)$ is not trivial. The example below shows that in general the knowledge of $\kappa(G)$ and $\lambda(G)$ is not enough to decide whether G is $(i, j) + \text{connected}$.

Example 3.3 For graphs on Fig. 1 we have $\kappa(G_1) = \kappa(G_2) = 2$ and $\lambda(G_1) = \lambda(G_2) = 3$. Both graphs are $(1, 0) + \text{connected} \implies (0, 1) + \text{connected}$, and $(0, 2) + \text{connected}$. Graph G_1 is not $(1, 1) + \text{connected}$, while graph G_2 is.

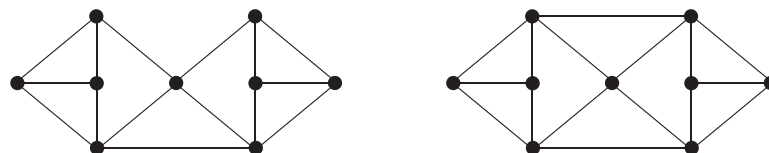


Fig. 1. Graphs G_1 and G_2 from Example 3.3.

Definition 3.4 Let G be a k -edge connected graph and $0 \leq a < k$. The a -edge fault diameter of G is

$$\mathcal{D}_a^E(G) = \max \{ \mathcal{D}(G \setminus X) \mid X \subseteq E(G), |X| = a \}.$$

Definition 3.5 Let G be a k -connected graph and $0 \leq a < k$. The a -fault diameter (or a -vertex fault diameter) of G is

$$\mathcal{D}_a^V(G) = \max \{ \mathcal{D}(G \setminus X) \mid X \subseteq V(G), |X| = a \}.$$

Note that $\mathcal{D}_a^E(G)$ is the largest diameter among the diameters of subgraphs of G with a edges deleted, and $\mathcal{D}_a^V(G)$ is the largest diameter over all subgraphs of G with a vertices deleted. In particular, $\mathcal{D}_0^E(G) = \mathcal{D}_0^V(G) = \mathcal{D}(G)$, the diameter of G . For $p \geq \kappa(G)$ and for $q \geq \lambda(G)$ we set $\mathcal{D}_p^V(G) = \infty$, $\mathcal{D}_q^E(G) = \infty$, as some of the subgraphs are not vertex connected or edge connected, respectively.

It is known [6] that for any connected graph G the inequalities below hold.

- (1) $\mathcal{D}(G) = \mathcal{D}_0^E(G) \leq \mathcal{D}_1^E(G) \leq \mathcal{D}_2^E(G) \leq \dots \leq \mathcal{D}_{\lambda(G)-1}^E(G) < \infty$.
- (2) $\mathcal{D}(G) = \mathcal{D}_0^V(G) \leq \mathcal{D}_1^V(G) \leq \mathcal{D}_2^V(G) \leq \dots \leq \mathcal{D}_{\kappa(G)-1}^V(G) < \infty$.

Definition 3.6 Let G be a (p, q) -connected graph. The (p, q) -mixed fault diameter of G is

$$\mathcal{D}_{(p,q)}(G) = \max \{ \mathcal{D}(G \setminus (X \cup Y)) \mid X \subseteq V(G), Y \subseteq E(G), |X| = p, |Y| = q \}.$$

Note that by Definition 3.6 the endpoints of edges of set Y can be in X . In this case we may get the same subgraph of G by deleting p vertices and fewer than q edges. It is however not difficult to see that the diameter of such subgraph is smaller than or equal to the diameter of some subgraph of G where exactly p vertices and exactly q edges are deleted. So the condition that the endpoints of edges of set Y are not in X need not to be included in Definition 3.6. The mixed fault diameter $\mathcal{D}_{(p,q)}(G)$ is the largest diameter among the diameters of all subgraphs obtained from G by deleting p vertices and q edges, hence $\mathcal{D}_{(0,0)}(G) = \mathcal{D}(G)$, $\mathcal{D}_{(0,a)}(G) = \mathcal{D}_a^E(G)$ and $\mathcal{D}_{(a,0)}(G) = \mathcal{D}_a^V(G)$.

Let $\mathcal{H}_a^V = \{G \setminus X \mid X \subseteq V(G), |X| = a\}$ and $\mathcal{H}_b^E = \{G \setminus X \mid X \subseteq E(G), |X| = b\}$. It is easy to see that

- (1) $\max \{ \mathcal{D}_b^E(H) \mid H \in \mathcal{H}_a^V \} = \mathcal{D}_{(a,b)}(G)$,
- (2) $\max \{ \mathcal{D}_a^V(H) \mid H \in \mathcal{H}_b^E \} = \mathcal{D}_{(a,b)}(G)$.

In previous work [6] on vertex, edge and mixed fault diameters of connected graphs the following theorem has been proved.

Theorem 3.7 *Let G be (p, q) -connected graph and $p > 0$.*

- *If $q > 0$, then $\mathcal{D}_{p+q}^E(G) \leq \mathcal{D}_{(1,p+q-1)}(G) \leq \dots \leq \mathcal{D}_{(p,q)}(G)$.*
- *If $q = 0$, then $\mathcal{D}_p^E(G) \leq \mathcal{D}_{(1,p-1)}(G) \leq \dots \leq \mathcal{D}_{(p-1,1)}(G) \leq \mathcal{D}_p^V(G) + 1$.*

Note that for $(p+1)$ -connected graph G , $p > 0$, we have either

$$\mathcal{D}_p^E(G) \leq \mathcal{D}_{(1,p-1)}(G) \leq \dots \leq \mathcal{D}_{(p-1,1)}(G) \leq \mathcal{D}_p^V(G)$$

or

$$\mathcal{D}_p^E(G) \leq \mathcal{D}_{(1,p-1)}(G) \leq \dots \leq \mathcal{D}_{(p-1,1)}(G) = \mathcal{D}_p^V(G) + 1.$$

For example, complete graphs, complete bipartite graphs, and cycles are graphs with $\mathcal{D}_{(p-1,1)}(G) = \mathcal{D}_p^V(G) + 1$ for all meaningful values of p . More examples of both types of graphs can be found in [6].

4 Fault diameters of Cartesian graph bundles

Cartesian graph bundles are a generalization of Cartesian graph products, first studied in [23,24]. Let G_1 and G_2 be graphs. The *Cartesian product* of graphs G_1 and G_2 , $G = G_1 \square G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$. Vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1 v_2 \in E(G_2)$ and $u_1 = u_2$. For further reading on graph products we recommend [17].

Definition 4.1 *Let B and F be graphs. A graph G is a Cartesian graph bundle with fibre F over the base graph B if there is a graph map $p : G \rightarrow B$ such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F , and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \square K_2$.*

More precisely, the mapping $p : G \rightarrow B$ maps graph elements of G to graph elements of B , i.e. $p : V(G) \cup E(G) \rightarrow V(B) \cup E(B)$. In particular, here we also assume that the vertices of G are mapped to vertices of B and the edges of G are mapped either to vertices or to edges of B . We say an edge $e \in E(G)$ is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*. The mapping p will also be called the *projection* (of the bundle G to its base B). Note that each edge $e = uv \in E(B)$ naturally induces an isomorphism $\varphi_e : p^{-1}(\{u\}) \rightarrow p^{-1}(\{v\})$ between two fibres. It may be interesting to note that while it is well-known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) [17], there may be many different graph bundle representations of the same graph [32]. Here we assume that the bundle representation is given. Note that in some cases finding a representation of G as a graph bundle can be found in polynomial time [19,30,31,32,33,34]. For example, one of the easy classes are the Cartesian

graph bundles over triangle-free base [19]. Note that a graph bundle over a tree T (as a base graph) with fibre F is isomorphic to the Cartesian product $T \square F$ (not difficult to see, appears already in [23]), i.e. we can assume that all isomorphisms φ_e are identities. For a later reference note that for any path $P \subseteq B$, $p^{-1}(P)$ is a Cartesian graph bundle over the path P , and one can define coordinates in the product $P \square F$ in a natural way.

In recent work on fault diameter of Cartesian graph products and bundles [2,3,4,5], analogous results were found for both fault diameter and edge fault diameter.

Theorem 4.2 [2] *Let F and B be k_F -connected and k_B -connected graphs, respectively, $0 \leq a < k_F$, $0 \leq b < k_B$, and G a Cartesian bundle with fibre F over the base graph B . Then*

$$\mathcal{D}_{a+b+1}^V(G) \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B) + 1.$$

Theorem 4.3 [5] *Let F and B be k_F -edge connected and k_B -edge connected graphs, respectively, $0 \leq a < k_F$, $0 \leq b < k_B$, and G a Cartesian bundle with fibre F over the base graph B . Then*

$$\mathcal{D}_{a+b+1}^E(G) \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B) + 1.$$

Before writing a theorem on bounds for the mixed fault diameter we recall a theorem on mixed connectivity.

Theorem 4.4 [14] *Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be (p_F, q_F) -connected and graph B be (p_B, q_B) -connected. Then Cartesian graph bundle G is $(p_F + p_B + 1, q_F + q_B)$ -connected.*

In recent work [15,16], an upper bound for the mixed fault diameter of Cartesian graph bundles, $\mathcal{D}_{(p+1,q)}(G)$, in terms of mixed fault diameter of the fibre and diameter of the base graph and in terms of diameter of the fibre and mixed fault diameter of the base graph, respectively, is given.

Theorem 4.5 [15] *Let G be a Cartesian graph bundle with fibre F over the base graph B , where graph F is (p, q) -connected, $p + q > 0$, and B is a connected graph with diameter $\mathcal{D}(B) > 1$. Then we have:*

- If $q > 0$, then $\mathcal{D}_{(p+1,q)}(G) \leq \mathcal{D}_{(p,q)}(F) + \mathcal{D}(B)$.
- If $q = 0$, then $\mathcal{D}_{p+1}^V(G) \leq \max\{\mathcal{D}_p^V(F), \mathcal{D}_{(p-1,1)}(F)\} + \mathcal{D}(B)$.

Theorem 4.5 improves results 4.2 and 4.3 for $a > 0$ and $b = 0$.

Let G be a Cartesian graph bundle with fibre F over the connected base graph B with diameter $\mathcal{D}(B) > 1$, and let $a > 0$. If graph F is $(a+1)$ -connected, i.e. $(a, 0)$ -connected, then by theorem 4.5 we have an upper bound for the vertex

fault diameter $\mathcal{D}_{a+1}^V(G) \leq \mathcal{D}_a^V(F) + \mathcal{D}(B) + 1$ for any graph F . Similarly, $\mathcal{D}_{a+1}^V(G) \leq \mathcal{D}_a^V(F) + \mathcal{D}(B)$ if $\mathcal{D}_{(a-1,1)}(F) \leq \mathcal{D}_a^V(F)$ holds.

If graph F is $(a+1)$ -edge connected, i.e. $(0, a)$ -connected, then by theorems 3.7 and 4.5 we have an upper bound for the edge fault diameter $\mathcal{D}_{a+1}^E(G) \leq \mathcal{D}_{(1,a)}(G) \leq \mathcal{D}_a^E(F) + \mathcal{D}(B)$.

Theorem 4.6 [16] *Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and graph B be (p, q) -connected, $p + q > 0$. Then we have:*

- If $q > 0$, then $\mathcal{D}_{(p+1,q)}(G) \leq \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$.
- If $q = 0$, then $\mathcal{D}_{p+1}^V(G) \leq \mathcal{D}(F) + \max\{\mathcal{D}_p^V(B), \mathcal{D}_{(p-1,1)}(B)\}$.

Theorem 4.6 improves results 4.2 and 4.3 for $a = 0$ and $b > 0$.

Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and let $b > 0$. If graph B is $(b+1)$ -connected, i.e. $(b, 0)$ -connected, then by Theorem 4.6 we have an upper bound for the vertex fault diameter $\mathcal{D}_{b+1}^V(G) \leq \mathcal{D}(F) + \mathcal{D}_b^V(B) + 1$ for any graph B . Similarly, $\mathcal{D}_{b+1}^V(G) \leq \mathcal{D}(F) + \mathcal{D}_b^V(B)$ if $\mathcal{D}_{(b-1,1)}(B) \leq \mathcal{D}_b^V(B)$ holds.

If graph B is $(b+1)$ -edge connected, i.e. $(0, b)$ -connected, then by theorems 3.7 and 4.5 we have an upper bound for the edge fault diameter $\mathcal{D}_{b+1}^E(G) \leq \mathcal{D}_{(1,b)}(G) \leq \mathcal{D}(F) + \mathcal{D}_b^E(B)$.

In the case when $a = b = 0$ the fault diameter is determined exactly.

Proposition 4.7 [15] *Let G be a Cartesian graph bundle with fibre F over the base graph B , and graphs F and B be connected graphs with diameters $\mathcal{D}(F) > 1$ and $\mathcal{D}(B) > 1$. Then*

$$\mathcal{D}_1^V(G) = \mathcal{D}_1^E(G) = \mathcal{D}(G) = \mathcal{D}(F) + \mathcal{D}(B).$$

In other words, the diameter of a nontrivial Cartesian graph bundle does not change when one element is faulty.

Here we improve results of theorems 4.2 and 4.3 for positive a and b .

5 The results - improved bounds

Before stating and proving the main theorems, we introduce some notation used in this section.

Let G be a Cartesian graph bundle with fibre F over the base graph B . The *fibre of vertex* $x \in V(G)$ is denoted by F_x , formally, $F_x = p^{-1}(\{p(x)\})$.

We will also use notation $F(u)$ for the fibre of the vertex $u \in V(B)$, i.e. $F(u) = p^{-1}(\{u\})$. Note that $F_x = F(p(x))$. We will also use shorter notation $x \in F(u)$ for $x \in V(F(u))$.

Let $u, v \in V(B)$ be distinct vertices, and Q be a path from u to v in B , and $x \in F(u)$. Then the *lift of the path Q to the vertex $x \in V(G)$* , \tilde{Q}_x , is the path from $x \in F(u)$ to a vertex in $F(v)$, such that $p(\tilde{Q}_x) = Q$ and $\ell(\tilde{Q}_x) = \ell(Q)$. Let $x, x' \in F(u)$. Then \tilde{Q}_x and $\tilde{Q}_{x'}$ have different endpoints in $F(v)$ and are disjoint paths if and only if $x \neq x'$. In fact, two lifts \tilde{Q}_x and $\tilde{Q}_{x'}$ are either disjoint $\tilde{Q}_x \cap \tilde{Q}_{x'} = \emptyset$ or equal, $\tilde{Q}_x = \tilde{Q}_{x'}$. We will also use notation \tilde{Q} for lifts of path Q to any vertex in $F(u)$.

Let Q be a path from u to v and $e = uw \in E(Q)$. We will use notation $Q \setminus e$ for the subpath from w to v , i.e. $Q \setminus e = Q \setminus \{u, e\} = Q \setminus \{u\}$.

Let G be a graph and $X \subseteq S(G)$ be a set of elements of G . A path P from a vertex x to a vertex y *avoids* X in G , if $S(P) \cap X = \emptyset$, and it *internally avoids* X , if $(S(P) \setminus \{x, y\}) \cap X = \emptyset$.

5.1 Vertex fault diameter of Cartesian graph bundles

Theorem 5.1 *Let G be a Cartesian graph bundle with fibre F over the base graph B , graphs F and B be k_F -connected and k_B -connected, respectively, and let $0 < a < k_F$, $0 < b < k_B$. If for fault diameters of graphs F and B , $\mathcal{D}_{(a-1,1)}^V(F) \leq \mathcal{D}_a^V(F)$ and $\mathcal{D}_{(b-1,1)}^V(B) \leq \mathcal{D}_b^V(B)$ hold then*

$$\mathcal{D}_{a+b+1}^V(G) \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B).$$

Proof. Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be $(a+1)$ -connected, $a > 0$, graph B be $(b+1)$ -connected, $b > 0$, and let $\mathcal{D}_{(a-1,1)}^V(F) \leq \mathcal{D}_a^V(F)$, $\mathcal{D}_{(b-1,1)}^V(B) \leq \mathcal{D}_b^V(B)$. Then $\mathcal{D}_a^V(F) \geq 2$, $\mathcal{D}_b^V(B) \geq 2$, and Cartesian bundle G is $(a+b+2)$ -connected. Let $X \subseteq V(G)$ be a set of faulty vertices, $|X| = a+b+1$, and let $x, y \in V(G) \setminus X$ be two distinct nonfaulty vertices in G . We shall consider the distance $d_{G \setminus X}(x, y)$.

- Suppose first that x and y are in the same fibre, i.e. $p(x) = p(y)$.
If $|X \cap V(F_x)| \leq a$, then $d_{G \setminus X}(x, y) \leq \mathcal{D}_a^V(F)$.
If $|X \cap V(F_x)| > a$, then outside of fibre F_x there are at most b faulty vertices. As graph B is $(b+1)$ -connected, there are at least $b+1$ neighbors of vertex $p(x)$ in B . Therefore there exist a neighbor v of vertex $p(x)$ in B , such that $|X \cap F(v)| = 0$, and there is a path $x \rightarrow x' \xrightarrow{P} y' \rightarrow y$, which avoids X , where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G \setminus X}(x, y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$.
- Now assume that x and y are in distinct fibres, i.e. $p(x) \neq p(y)$. Let $X_B = \{v \in V(B) \setminus \{p(x), p(y)\}; |X \cap F(v)| > 0\}$. We distinguish two cases.

- (1) If $|X_B| \geq b$, then let $X'_B \subseteq X_B$ be an arbitrary subset of X_B with $|X'_B| = b$. The subgraph $B \setminus X'_B$ is a connected graph and there exists a path Q in $B \setminus X'_B$ from $p(x)$ to $p(y)$ with $\ell(Q) \leq \mathcal{D}_b^V(B)$. In $p^{-1}(Q) = F \square Q$ there are at most $a + 1$ faulty vertices. Let $x' \in F_y$ be the endpoint of the path \tilde{Q}_x , the lift of Q . We distinguish two cases.

- (a) If $x' = y$, then \tilde{Q}_x is a path from x to y in G . If \tilde{Q}_x avoids X , then $d_{G \setminus X}(x, y) \leq \ell(Q) \leq \mathcal{D}_b^V(B)$. If \tilde{Q}_x does not avoid X , then there are at most a faulty vertices outside of the path \tilde{Q}_x in $F \square Q$. As the graph F is $(a + 1)$ -connected, there are at least $a + 1$ neighbors of x in F_x . Since there are more neighbors than faulty vertices (outside of \tilde{Q}_x in $F \square Q$), there exists a neighbor $v \in V(F_x)$ of x , such that the lift \tilde{Q}_v avoids X . The endpoint of the path \tilde{Q}_v in fibre F_y is a neighbor of y , therefore $d_{G \setminus X}(x, y) \leq 1 + \ell(Q) + 1 \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$.
- (b) Let $x' \neq y$. If $|V(F_x) \cap X| = a + 1$ or $|V(F_y) \cap X| = a + 1$, then obviously $d_{G \setminus X}(x, y) \leq \ell(Q) + \mathcal{D}(F) \leq \mathcal{D}_b^V(B) + \mathcal{D}_a^V(F)$.

Now assume $|V(F_x) \cap X| \leq a$ and $|V(F_y) \cap X| \leq a$. If \tilde{Q}_x or \tilde{Q}_y avoids X , then $d_{G \setminus X}(x, y) \leq \ell(Q) + \mathcal{D}_a^V(F) \leq \mathcal{D}_b^V(B) + \mathcal{D}_a^V(F)$. Suppose that paths \tilde{Q}_x and \tilde{Q}_y do not avoid X . Then there are at most $a - 1$ faulty vertices outside of paths \tilde{Q}_x and \tilde{Q}_y in $F \square Q$. Let $X' \subseteq V(F_y)$ be defined as $X' = \{v \in V(F_y) \setminus \{x', y\}, |\tilde{Q}_v \cap X| > 0\}$. Then $|X'| \leq a - 1$. There is a path P from x' to y in $F_y \setminus X'$ of length $\ell(P) \leq \mathcal{D}_{a-1}^V(F) \leq \mathcal{D}_a^V(F)$. Note that the path P internally avoids X . If x' and y are not adjacent, then $\ell(P) \geq 2$. For the neighbor v' of x' on the path P , $e' = x'v' \subset P$, the lift $\tilde{Q}_{v'}$ avoids X . Let $v \in V(F_x)$ be the endpoint of the lift $\tilde{Q}_{v'}$. Then the path $x \rightarrow v \xrightarrow{\tilde{Q}} v' \xrightarrow{P \setminus e'} y$ avoids X , therefore $d_{G \setminus X}(x, y) \leq 1 + \ell(Q) + \mathcal{D}_a^V(F) - 1 \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$. If x' and y are adjacent, then remove from F_y the set of vertices X' and the edge $e = x'y$. There is a path P' from x' to y in $F_y \setminus (X' \cup \{e\})$ of length $2 \leq \ell(P') \leq \mathcal{D}_{(a-1,1)}(F)$, that internally avoids X . As before, for the neighbor w' of x' on the path P' the lift $\tilde{Q}_{w'}$ avoids X . Therefore $d_{G \setminus X}(x, y) \leq 1 + \ell(Q) + \mathcal{D}_{(a-1,1)}(F) - 1 \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$.

- (2) If $|X_B| < b$, then the subgraph $B \setminus X_B$ is (at least) 2-connected, thus also 2-edge connected. If the vertex $p(y)$ is not a neighbor of $p(x)$, then there is a path Q from $p(x)$ to $p(y)$ in B with $2 \leq \ell(Q) \leq \mathcal{D}_{b-1}^V(B) \leq \mathcal{D}_b^V(B)$ that internally avoids X_B . Let $v \in V(Q)$ be a neighbor of $p(x)$, $e' = p(x)v$. Then there is a path $x \rightarrow x' \xrightarrow{P} y' \xrightarrow{\tilde{Q} \setminus e'} y$, which avoids X , where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G \setminus X}(x, y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_b^V(B) - 1 \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$. If $e = p(x)p(y) \in E(B)$, then $B \setminus (X_B \cup \{e\})$ is a connected graph and there is a path Q' from $p(x)$ to $p(y)$ with $2 \leq \ell(Q') \leq \mathcal{D}_{(b-1,1)}(B)$ that internally avoids X_B . Similarly as before we have $d_{G \setminus X}(x, y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_{(b-1,1)}(B) - 1 \leq \mathcal{D}_a^V(F) + \mathcal{D}_b^V(B)$. \square

Theorem 5.1 improves Theorem 4.2 on the class of Cartesian graph bundles for which both, the fiber and the base graph, are at least 2-connected. Theorem 5.1 also improves result of [27] on the Cartesian graph products with at least 2-connected factors. The next example shows that the bound of Theorem 5.1 is tight.

Example 5.2 Let $F = B = K_4 \setminus \{e\}$. Then graph F is 2-connected and $\mathcal{D}_1^E(F) = \mathcal{D}_1^V(F) = 2$. The vertex fault diameter of Cartesian graph product $F \square F$ on Fig. 2 is $\mathcal{D}_3^V(F \square F) = \mathcal{D}_1^V(F) + \mathcal{D}_1^V(F) = 4$.

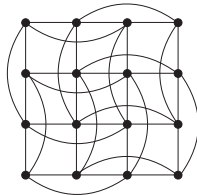


Fig. 2. Cartesian graph product of two factors $K_4 \setminus \{e\}$.

Example 5.3 Cycle C_4 is 2-connected graph and $\mathcal{D}_1^E(C_4) = \mathcal{D}_1^V(C_4) + 1 = 3$. The vertex fault diameter of Cartesian graph bundle G with fibre C_4 over base graph C_4 on Fig. 3 is $\mathcal{D}_3^V(G) = \mathcal{D}_1^V(C_4) + \mathcal{D}_1^V(C_4) + 1 = 5$.

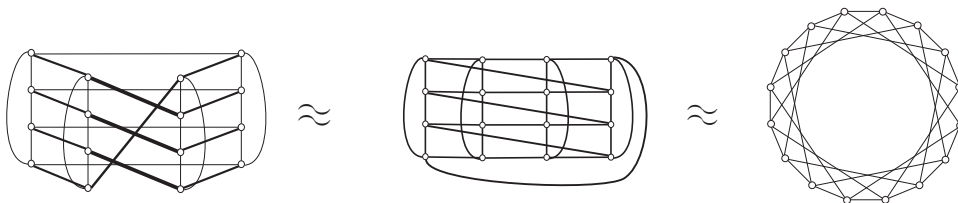


Fig. 3. Twisted torus: Cartesian graph bundle with fibre C_4 over base C_4 .

It is less known that graph bundles also appear as computer topologies. A well known example is the twisted torus on Fig. 3. Cartesian graph bundle with fibre C_4 over base C_4 is the ILLIAC IV architecture [7], a famous supercomputer that inspired some modern multicomputer architectures. It may be interesting to note that the original design was a graph bundle with fibre C_8 over base C_8 , but due to high cost a smaller version was build [35].

5.2 Edge fault diameter of Cartesian graph bundles

Let G be a k -edge connected graph and $0 \leq a < k$. Note that if $a > 0$ then $\mathcal{D}_a^E(G) \geq 2$ for any graph G . More precisely, $\mathcal{D}_a^E(G) \geq 2$ if $a > 0$ or ($a = 0$ and G is not a complete graph). Furthermore, $\mathcal{D}_a^E(G) = 1$ if and only if $a = 0$ and G is a complete graph.

Theorem 5.4 *Let G be a Cartesian graph bundle with fibre F over the base graph B , graphs F and B be k_F -edge connected and k_B -edge connected, respectively, and let $0 \leq a < k_F$, $0 \leq b < k_B$. If for edge fault diameters of graphs F and B , $\mathcal{D}_a^E(F) \geq 2$ and $\mathcal{D}_b^E(B) \geq 2$ hold then*

$$\mathcal{D}_{a+b+1}^E(G) \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B).$$

Proof. Let G be a Cartesian graph bundle with fibre F over the base graph B , the graph F be $(a+1)$ -edge connected, $\mathcal{D}_a^E(F) \geq 2$, and the graph B be $(b+1)$ -edge connected, $\mathcal{D}_b^E(B) \geq 2$. Then the Cartesian bundle G is $(a+b+2)$ -edge connected. Let $Y \subseteq E(G)$ be the set of faulty edges, $|Y| = a+b+1$. Denote the set of degenerate edges in Y by Y_D , and the set of nondegenerate edges by Y_N , $Y = Y_N \cup Y_D$, $p(Y_D) \subseteq V(B)$, $p(Y_N) \subseteq E(B)$. Let $x, y \in V(G)$ be two arbitrary distinct vertices in G . We shall consider the distance $d_{G \setminus Y}(x, y)$.

- Suppose first that x and y are in the same fibre, i.e. $p(x) = p(y)$.
 If $|Y_D \cap E(F_x)| \leq a$, then $d_{G \setminus Y}(x, y) \leq \mathcal{D}_a^E(F)$.
 If $|Y_D \cap E(F_x)| > a$, then outside of the fibre F_x there are at most b faulty edges. As the graph B is $(b+1)$ -edge connected, there are at least $b+1$ neighbors of the vertex $p(x)$ in B . Therefore there exist a neighbor v of vertex $p(x)$ in B , $e = p(x)v \in E(B)$, such that $|Y_D \cap F(v)| = 0$ and $p(e) \notin p(Y_N)$, and hence there is a path $x \rightarrow x' \xrightarrow{P} y' \rightarrow y$, which avoids Y , where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G \setminus Y}(x, y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B)$.
- Now assume that x and y are in distinct fibres, i.e. $p(x) \neq p(y)$. We distinguish two cases.
 - (1) If $|Y_N| \geq b$, then let $Y'_N \subseteq Y_N$ be an arbitrary subset of Y_N with $|Y'_N| = b$. The subgraph $B \setminus p(Y'_N)$ is a connected graph and there exists a path Q from $p(x)$ to $p(y)$ with $\ell(Q) \leq \mathcal{D}_b^E(B)$. In $p^{-1}(Q) = F \square Q$ there are at most $a+1$ faulty edges. Let $x' \in F_y$ be the endpoint of the path \tilde{Q}_x , the lift of Q . We distinguish two cases.
 - (a) If $x' = y$, then \tilde{Q}_x is a path from x to y in G . If \tilde{Q}_x avoids Y , then $d_{G \setminus Y}(x, y) \leq \ell(Q) \leq \mathcal{D}_b^E(B)$. If \tilde{Q}_x does not avoid Y , then there are at most a faulty edges outside of the path \tilde{Q}_x in $F \square Q$. As the graph F is $(a+1)$ -edge connected, there are at least $a+1$ neighbors of x in F_x . Since there are more neighbors than faulty edges (outside of \tilde{Q}_x in $F \square Q$) there exist a neighbor $s \in V(F_x)$ of x , such that the path $x \rightarrow s \xrightarrow{\tilde{Q}} s' \rightarrow y$ avoids Y , where $s' \in V(F_y)$ is a neighbor of y , therefore $d_{G \setminus Y}(x, y) \leq 1 + \ell(Q) + 1 \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B)$.
 - (b) Let $x' \neq y$. If $|E(F_x) \cap Y| = a+1$ or $|E(F_y) \cap Y| = a+1$, then obviously $d_{G \setminus Y}(x, y) \leq \ell(Q) + \mathcal{D}(F) \leq \mathcal{D}_b^E(B) + \mathcal{D}_a^E(F)$.
 Now let $|E(F_x) \cap Y| \leq a$ and $|E(F_y) \cap Y| \leq a$. If \tilde{Q}_x or \tilde{Q}_y avoid Y , then $d_{G \setminus Y}(x, y) \leq \ell(Q) + \mathcal{D}_a^E(F) \leq \mathcal{D}_b^E(B) + \mathcal{D}_a^E(F)$. Suppose that paths \tilde{Q}_x and \tilde{Q}_y do not avoid Y . Then there are at most $a-1$

faulty edges outside of paths \tilde{Q}_x and \tilde{Q}_y in $F \square Q$. Let $Y'_D \subseteq E(F_y)$ be set of edges from x' to such neighbors $v'_i \in V(F_y)$ for which the paths $v'_i \xrightarrow{\tilde{Q}} v_i \rightarrow x$ do not avoid faulty edges, $Y'_D = \{e = x'v' \in E(F_y); |(\tilde{Q}'_v \cup vx) \cap Y| > 0, v = \tilde{Q}'_v \cap F_x\}$. Note that if x' is a neighbor of y then $x'y \in Y'_D$. Then the subgraph $F_y \setminus (Y'_D \cup Y_D)$ is a connected graph as there are at most $a + 1$ faulty edges in $p^{-1}(Q) = F \square Q$ and \tilde{Q}_x does not avoid Y . Therefore there is a path P from x' to y in $F_y \setminus (Y'_D \cup Y_D)$ of length $2 \leq \ell(P) \leq \mathcal{D}_a^E(F)$, which avoids Y and for the neighbor v' of x' on the path P , the lift $\tilde{Q}_{v'}$ avoids Y . Let $v = \tilde{Q}_{v'} \cap F_x$. Then $vx \notin Y$, and the path $x \rightarrow v \xrightarrow{\tilde{Q}} v' \xrightarrow{P \setminus e'} y$ avoids Y , thus $d_{G \setminus Y}(x, y) \leq 1 + \ell(Q) + \ell(P) - 1 \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B)$.

- (2) If $|Y_N| < b$, then there is a path Q from $p(x)$ to $p(y)$ in B which avoids $p(Y_N)$ of length $\ell(Q) \leq \mathcal{D}_{b-1}^E(B) \leq \mathcal{D}_b^E(B)$. If $|E(F_x) \cap Y_D| \leq a$ or $|E(F_y) \cap Y| \leq a$, then obviously $d_{G \setminus Y}(x, y) \leq \ell(Q) + \mathcal{D}_a^E(F) \leq \mathcal{D}_b^E(B) + \mathcal{D}_a^E(F)$.

Now let $|E(F_x) \cap Y| > a$ and $|E(F_y) \cap Y| > a$. Then outside of the fibres F_x and F_y there are at most $b - 1$ faulty edges. Let $Y'_N \subseteq E(B)$ be set of edges from $p(x)$ to such neighbors $v_i \in V(B)$ for which fibre $F(v_i)$ contains faulty edges, $Y'_N = \{e = p(x)v \in E(B); |F(v) \cap Y_D| > 0\}$. Note that if $p(x)$ is a neighbor of $p(y)$ then $p(x)p(y) \in Y'_N$. Then the subgraph $B \setminus (Y'_N \cup p(Y_N))$ is a connected graph as there are at most $b - 1$ faulty edges outside of fibres F_x and F_y . Therefore there is a path Q' from $p(x)$ to $p(y)$ with $2 \leq \ell(Q') \leq \mathcal{D}_b^E(B)$ that avoids $p(Y_N)$ and there is no faulty edges in the fibre $F(v)$ of the neighbor v of $p(x)$ on the path Q' , $e = p(x)v \subset Q'$. Hence there is a path $x \rightarrow x' \xrightarrow{P} y' \xrightarrow{\tilde{Q} \setminus e} y$, which avoids Y , where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G \setminus Y}(x, y) \leq 1 + \mathcal{D}(F) + \ell(Q) - 1 \leq \mathcal{D}_a^E(F) + \mathcal{D}_b^E(B)$. \square

Clearly, Theorem 5.4 improves Theorem 4.3 for all cases except in the following two cases: either when $a = 0$ and F is a complete graph or when $b = 0$ and B is a complete graph.

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